

**Assignment 3.**

This homework is due *Tuesday*, September 17.

There are total 30 points in this assignment. 24 points is considered 100%. If you go over 24 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 2.1–2.2 in Bartle–Sherbert.

## 1. QUICK CHEAT-SHEET

REMINDER. (Subsection 2.1.1) On the set  $\mathbb{R}$  of real numbers there two binary operations, denoted by  $+$  and  $\cdot$  and called addition and multiplication, respectively. These operations satisfy the following properties:

- (A1)  $a + b = b + a$  for all  $a, b \in \mathbb{R}$ ,
- (A2)  $(a + b) + c = a + (b + c)$  for all  $a, b, c \in \mathbb{R}$ ,
- (A3) there exists  $0 \in \mathbb{R}$  s.t.  $0 + a = a + 0 = a$  for all  $a \in \mathbb{R}$ ,
- (A4) for each  $a \in \mathbb{R}$  there exists an element  $-a$  s.t.  $a + (-a) = (-a) + a = 0$ ,
- (M1)  $ab = ba$  for all  $a, b \in \mathbb{R}$ ,
- (M2)  $(ab)c = a(bc)$  for all  $a, b, c \in \mathbb{R}$ ,
- (M3) there exists  $1 \in \mathbb{R}$  s.t.  $1 \cdot a = a \cdot 1 = a$  for all  $a \in \mathbb{R}$ ,
- (M4) for each  $a \neq 0$  in  $\mathbb{R}$  there exists an element  $1/a$  s.t.  $a \cdot (1/a) = (1/a) \cdot a = 1$ ,
- (D)  $a(b + c) = ab + ac$  and  $(b + c)a = ba + ca$  for all  $a, b, c \in \mathbb{R}$ .

REMINDER. (Subsection 2.1.5 in textbook) Let  $\mathbb{A}$  be a set with two operations  $+$  and  $\cdot$  satisfying A1–A4, M1–M3 and D (for example,  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ ). The set  $\mathbb{P} \subset \mathbb{A}$  is called the set of *positive elements* if

- (i) If  $a, b \in \mathbb{P}$ , then  $a + b \in \mathbb{P}$ ,
- (ii) If  $a, b \in \mathbb{P}$ , then  $ab \in \mathbb{P}$ ,
- (iii) If  $a \in \mathbb{A}$ , then exactly one of the following holds:  $a \in \mathbb{P}$ ,  $a = 0$ ,  $-a \in \mathbb{P}$ .

Then  $a < b$  if and only if  $b - a \in \mathbb{P}$ ;  $a \leq b$  if and only if  $b - a \in \mathbb{P} \cup 0$ .

One can prove the following (Theorem 2.1.7 in textbook):

THEOREM. Let  $a, b, c \in \mathbb{A}$ .

- (a) If  $a > b$  and  $b > c$  then  $a > c$ ,
- (b) if  $a > b$ , then  $a + c > b + c$ ,
- (c) if  $a > b$ ,  $c > 0$ , then  $ca > cb$ ,  
if  $a > b$ ,  $c < 0$ , then  $ca < cb$ .

## 2. EXERCISES

- (1) [4pt] (Parts of 2.1.1, 2, 5) For  $a, b \in \mathbb{R}$ , prove that
  - (a)  $-(a + b) = -a + (-b)$ , (*Hint*: show that the number  $-a + (-b)$  satisfies the definition of a number opposite to  $(a + b)$  in (A4).)
  - (b)  $-(-a) = a$ ,
  - (c) if  $a \neq 0$ ,  $b \neq 0$ , then  $1/(ab) = (1/a)(1/b)$ .

Every equality you write in Problems 1, 2 should be accompanied by a reference to the exact property of real numbers or theorem you are using.

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- (2) [2pt] Prove that if  $a \in \mathbb{R}$  satisfies  $a \cdot a = a$ , then  $a = 0$  or  $a = 1$ .
- (3) [5pt]
- ( $\sim$ Ex. 2.1.8a) Let  $x, y$  be rational numbers. Prove that  $xy, x - y$  are rational numbers.
  - (Ex. 2.1.8b) Let  $x$  be a rational number,  $y$  an irrational number. Prove that  $x + y$  is irrational. Prove that if, additionally,  $x \neq 0$ , then  $xy$  is irrational.
  - Let  $x, y$  be irrational numbers. Is it true that  $xy$  is always irrational? Is it true that  $xy$  is always rational?
- (4) [2pt] Prove that there does not exist a rational number  $r$  such that  $r^2 = 3$ .
- (5) [3pt](2.1.10) For  $a, b, c, d \in \mathbb{R}$ , prove that
- if  $a < b, c \leq d$ , then  $a + c < b + d$ ,
  - if  $0 < a < b, 0 < c \leq d$ , then  $0 < ac < bd$ ,
- Every inequality you write in Problems 5, 6 should be accompanied by a reference to the exact property (i)–(iii), Theorem above or a previously proved claim that you are using.
- (6) [2pt] (2.1.12+) Let  $a, b, c, d \in \mathbb{R}$  satisfy  $0 < a < b$  and  $c < d < 0$ . Give an example where  $ac < bd$ , an example where  $ac > bd$ , and an example where  $ac = bd$ .
- (7) In each case below, determine if  $P$  is a set of positive elements (i.e. whether it satisfies (i), (ii) and (iii)).
- [1pt]  $\mathbb{A} = \mathbb{Z}, P = \mathbb{N}$ ,
  - [1pt]  $\mathbb{A} = \mathbb{Z}, P = -\mathbb{N}$ ,
  - [1pt]  $\mathbb{A} = \mathbb{Q}, P = \{r \in \mathbb{Q} : r > 1\}$ ,
  - [1pt]  $\mathbb{A} = \mathbb{Q}, P = \{r \in \mathbb{Q} : r > -1\}$ ,
  - [1pt]  $\mathbb{A} = \mathbb{C}, P = \{z = x + iy \in \mathbb{C} : x > 0\}$ ,
  - [2pt] Prove that for  $\mathbb{A} = \mathbb{C}$ , there is no set of positive elements. (In other words, one cannot imbue  $\mathbb{C}$  with a meaningful order.)
- Note: items 7e, 7f deal with complex numbers that not everyone may be familiar with. So, these two questions are excluded from denominator of the grade for this homework. They are nevertheless included in numerator and you are encouraged to attempt them.
- (8) [5pt] ( $\sim$ Ex. 2.2.14bd,15bd) Determine and sketch the set of pairs  $(x, y)$  in  $\mathbb{R} \times \mathbb{R}$  that satisfy
- $|x| + |y| = 2$ .
  - $|x| - |y| = 2$ .
  - $|x| + |y| \geq 2$ .
  - $|x| - |y| \geq 2$ .